Improving Power Flow Convergence By Leveraging Region of Attraction Visualization

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***Abstract*—This paper both highlights the ease with which the Newton-Raphson power flow can converge to undesired alternative solutions and then presents techniques that can be used to improve power flow convergence by leveraging insights gained from visualizing the power flow solution region of attraction. The paper shows that, particularly for large-scale power systems, the common "flat start" initial voltage guess of one per unit with an angle of zero may not be in the region of attraction (ROA) for the desired power flow solution. Computationally tractable solution options are presented to reduce the likelihood of the power flow converging to an undesired solution. Results are demonstrated on power systems ranging in size from two to 23,600 buses.**

***Index Terms*—power flow convergence, power system visualization, power flow region of attraction, alternative power flow solutions**

I. INTRODUCTION

Power flow (PF) analysis is a fundamental tool in power system operation and planning. It is used to solve the set of nonlinear power balance equations to determine the voltage magnitudes and analysis at all the buses in an electric power system. These voltage values can then be used to determine how the power flows through the grid and to calculate the reactive power outputs of the generators. For at least 50 years, it has been known that the power flow can have multiple solutions [1]. Usually, but not always, the solution with the highest per-unit voltage magnitudes represents the solution at which the power system would actually operate and hence is sometimes known as the "operable solution" (OprS). Since this solution would be stable if typical power system dynamic models are included, it is also sometimes called the stable equilibrium point (SEP). The other solutions usually have lower voltages and are known as either low-voltage solutions or alternative solutions [2], [3]. Here, the abbreviation AltS is used to denote a particular alternative solution. Some authors also refer to them as unstable equilibrium points (UEPs) since they would be unstable with typical power grid dynamics.

By far the most common technique for solving the power flow solutions is to use the Newton-Raphson (NR) approach [4]. In the NR, the non-linear equations are solved by making an initial guess of the bus voltages and then iterating until either convergence, divergence, or a maximum number of iterations is reached. Particularly for small power systems, the common practice is to make an initial voltage guess in which all voltage magnitudes are one per unit and all angles are zero. This is known as a "flat start" solution. Whether the NR converges to the operable solution or another solution depends on this initial guess.

Sometimes one or more of the AltSs can be desired, such as in some voltage stability proximity algorithms [5], [6], [7]. As such, techniques have been developed specifically for converging to these solutions [2], [8], [9], [10]. Yet for the vast majority of power flows, the goal is to converge only to the OprS. Usually, this is achieved either by using the flat start for small systems or by starting from a previous solution for large systems. Then, for example, if new buses have been added, initial voltage guesses are only needed for this subset of buses with flat start values a common choice. However, as is presented here, this can result in convergence to AltS.

The contribution of this paper is in the area of gaining insight into power flow convergence and then leveraging this to present improved techniques for increasing the likelihood of convergence to the desired OprS. This paper is organized as follows. Section II provides necessary background information on the power flow and visualization of its regions of attraction. Section III then introduces the test power flow cases. Section IV then presents and demonstrates the techniques for improved convergence, while Section V concludes with recommendations and directions for future work. All the computational results presented here have been done using PowerWorld Simulator version 24 [11].

II. BACKGROUND

The power flow solves the bus power balance equations to determine the phasor voltages at each bus in an electric grid. Since the phasor voltages and complex impedances can be expressed in either polar or rectangular form, there are several formulations of these equations, with the selected form impacting the NR convergence. Here, two forms are used: the polar form from [12] (NRP) with the voltages given in polar notation (i.e., the bus k voltage is $V_k = V_k \angle \theta_k$) and the rectangular form from [13] (NRR) with the voltages in rectangular notation ($V_k = e_k + jf_k$). For a grid with N buses, the real and reactive NRP power balance equations are given in (1) and (2), while the NRR equations are in (3) and (4),

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$$P_{Gk} - P_{Lk}(V_k) = V_k \sum_{n=1}^{N} V_n \left[G_{kn} \cos(\theta_k - \theta_n) + B_{kn} \sin(\theta_k - \theta_n) \right] \quad (1)$$

$$Q_{Gk} - Q_{Lk}(V_k) = V_k \sum_{n=1}^{N} V_n \left[G_{kn} \sin(\theta_k - \theta_n) - B_{kn} \sin(\theta_k - \theta_n) \right] \quad (2)$$

$$P_{Gk} - P_{Lk}(V_k) = \sum_{n=1}^{N} \left[e_k \left(e_n G_{kn} + f_n B_{kn} \right) + f_k \left(f_n G_{kn} - e_n B_{kn} \right) \right] \quad (3)$$

$$Q_{Gk} - Q_{Lk}(V_k) = \sum_{n=1}^{N} \left[f_k \left(e_n G_{kn} + f_n B_{kn} \right) - e_k \left(f_n G_{kn} - e_n B_{kn} \right) \right] \quad (4)$$

where P_{Gk} and Q_{Gk} are respectively the real and reactive power generation at bus k, $P_{Lk}(V_k)$ and $Q_{Lk}(V_k)$ are the loads; G_{kn} and B_{kn} are the real and imaginary components of the bus admittance matrix. Note, this formulation allows the load values to be a function of their bus voltage magnitudes. With the common ZIP load model [14], the bus load is represented as having a portion that varies with the square of the voltage magnitude, a portion that varies linearly with voltage magnitude, and then a constant power component.

Of course, in a typical power flow, not all the voltages are independent variables. For example, at the slack bus, the voltage magnitude and angle are fixed, and at generator (PV) buses, the voltage magnitude is fixed, within reactive power limits. Additionally, as is the case here, various external actions can be imposed on the power flow solution, including checking generator reactive power limits, adjusting transformer taps and phase angles, changing the statuses of switched shunt devices, and modifying generation to enforce area interchange constraints. All of these can impact the overall power flow convergence.

With an N bus grid, the number of independent voltage variables will be between N and 2N. Define this value as m, and the set of initial voltage guesses as $\mathbf{x}^0 \in \Re^m$ (represented in either polar or rectangular format). The region of attraction (ROA) (also known as the domain of attraction) for an NR algorithm is the set of voltage guesses that converge to a particular solution. Papers visualizing the power flow ROA include [3], [15], [16], and [17], with several noting the ROA's fractal nature, at least with some power flow formulations. These fractal ROAs are also mentioned in [18], which highlights the need for further research on power flow convergence.

While several prior works have done a nice job of visualizing the ROA for electric grids with two or three buses, and [17] considers a grid with 14 buses, there is very little work looking at large-scale grids. This is not surprising given that the ROA is an m-dimensional space, and that calculating the values needed to show even for a 2D projection of the ROA can be computationally involved for larger-scale grids. The contribution of this paper is to address this shortcoming and leverage the insights that can be gained from the ROA visualizations to provide techniques for improved power flow convergence.

III. Test Cases and Initial ROA Visualizations

As a starting point, consider the two-bus grid shown in Figure 1 with its OprS shown; in the figure, the green arrows are used to visualize the flow of real power and the blue arrows the flow of reactive power. Figure 2 shows the same grid, except that it is solved with an initial voltage guess that results in convergence to an AltS. Note the high reactive power output from the slack generator, indicating this would not correspond to an operable solution. Nevertheless, it is a solution since it satisfies the power balance equations.

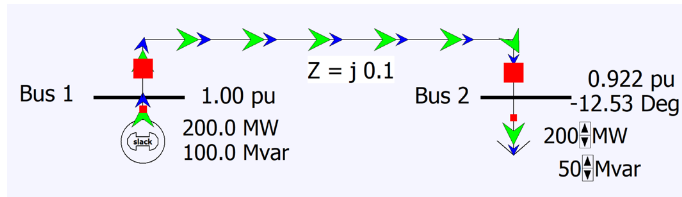

Figure 1. Two-Bus Electric Grid, Operable Solution (OprS)

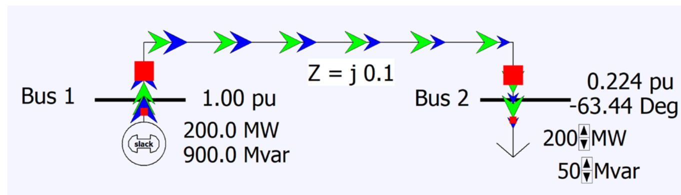

Figure 2. Two-Bus Electric Grid, Alternative Solution (AltS)

The ROA is then visualized by repeatedly solving the power flow with varied initial voltage guesses and then using a color mapping to differentiate between the guess that solved to the OprS and those that solved to the AltS. This is shown in Figure 3, with orange shades indicating guesses that resulted in the OprS and purple guesses solving to the AltS using the NRP algorithm; the number of iterations needed to get each solution is indicated by the color intensity, with the iteration count set to negative if it solved to the AltS. The x-axis shows the voltage angle guesses, ranging from -180 to 180 degrees in one-degree increments, and the y-axis shows the voltage magnitude guesses, ranging from 0 to 1.5 pu in 0.01 pu increments. Hence, generating the figure requires a total of 54,000 power flow solutions (360 times 150). Of this total, approximately 20.3% converged to the OprS with an average of 4.8 iterations, and 79.7% converged to the AltS with an average of 7.2 iterations.

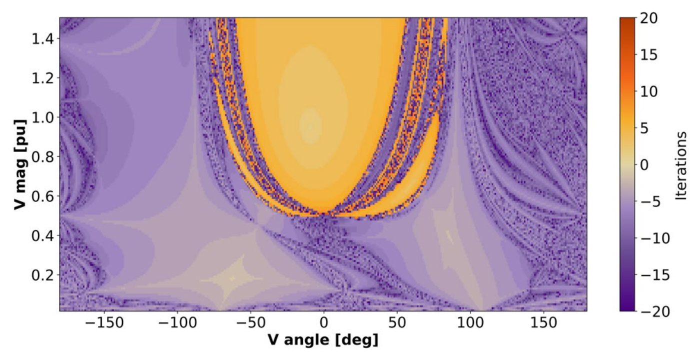

Figure 3. Two-Bus NRP Region of Attraction (ROA)Visualization

Given that the goal here is to improve power flow convergence for realistic grids to the AltS by leveraging ROA insights, larger grids also need to be considered. An initial step in this direction is the 42-bus case from [19], whose one-line is shown in Figure 4, with the figure showing the voltage magnitude contour [20] for the OprS. At this solution, the voltage angle variation across the system ranges from zero degrees at Bus 4 (Grafton345, the slack bus) to -40.9° at Bus 34 (Rose138).

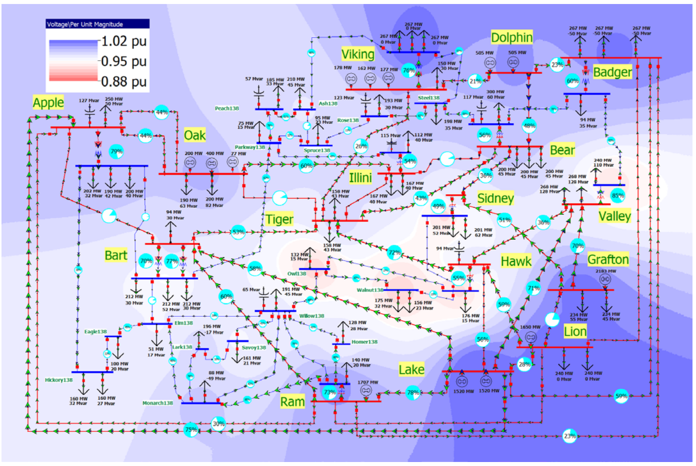

Figure 4. 42-Bus Electric Grid, Operable Solution (OprS)

To gain convergence insight again the ROA is considered. However, recognizing that the ROA is an m-dimensional space now with m approximately 80, only a 2D projection of the full ROA can be shown in a figure. Nevertheless, as will be leveraged in the next section, this can still provide good insight. One approach is just to vary the voltage magnitude and angle guess at a single bus, keeping all the other voltages constant at the OprS. An example of this is shown in Figure 5, where the voltage varies at Bus 34, again using the NRP algorithm. Of course, any bus could be chosen, and actually, all buses will be considered in the next section. The rationale for selecting Bus 34 initially is that it has the largest angle variation from the slack bus. What is particularly interesting from Figure 5 is that the flat start voltage of $1.0\angle 0°$ is not in the ROA for the OprS. Instead, it converges to the AltS shown in Figure 6, in which the Bus 34 voltage is 0.61 pu (note the drastic change in the contour color key compared with Figure 4). Here, only 1.0% of the guesses converge to OprS (with an average of 6.5 iterations) and only 2.7% to the AltS (with an average of 16 iterations); the remainder diverge. Given that when an engineer adds a new bus to an existing system, it is common to just set the new bus voltage to its flat start value, this is concerning.

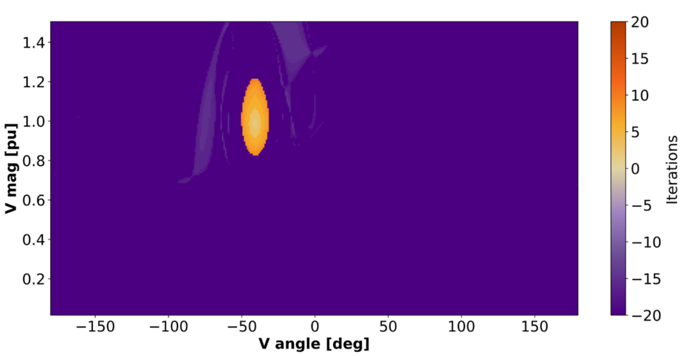

Figure 5. 42-Bus Electric Grid 2D ROA Projection for Bus 34

The last example is a 23,600 bus synthetic electric grid from [21] covering the central portion of the US, with Figure 7 showing its one-line along with a contour of the bus voltage angles (in degrees) at its OprS. Note the variation of almost 360° across the grid. Both this grid and the 42-bus grid are available for public download at the repository described in [22].

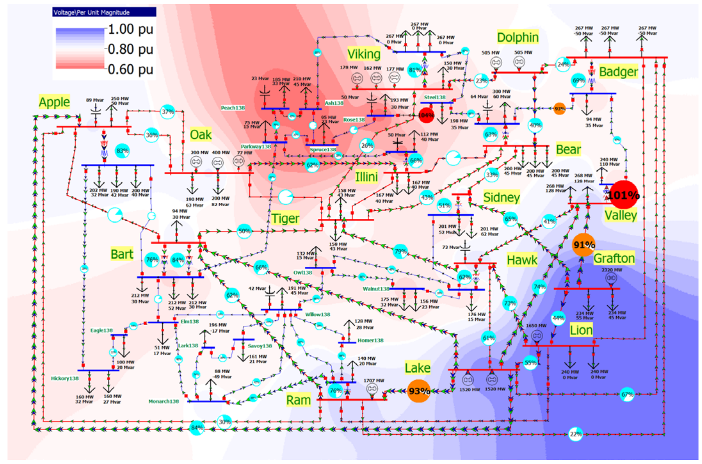

Figure 6. 42-Bus Electric Grid, An Alternative Solution (AltS)

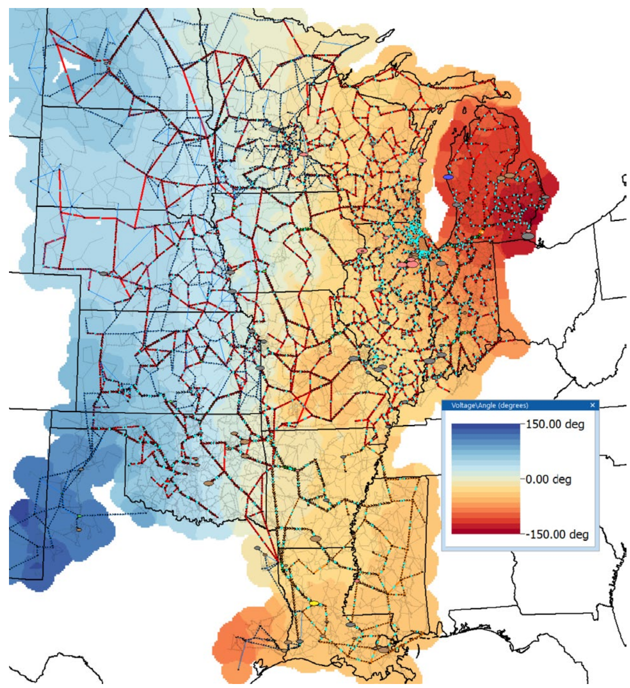

Figure 7. 23,600-Bus Electric Grid Voltage Angle Contour

In this case, the ROA has an extremely high dimension; however, a 2D projection can be created by selecting a particular bus. However, aside from this now being more computationally taxing (i.e., solving tens of thousands of power flows with a 23,600-bus grid), given that the goal is to solve for the OprS, only voltage magnitude guesses close to 1.0 per unit need to be considered. If the voltage magnitude guess is fixed at 1.0 per unit, then the only independent variable is the voltage angle, allowing the results to be shown in a graph. An example of this is shown in Figure 8 for Bus 180977 (located in the US state of Michigan), which has OprS voltage of $1.013\angle -116.6°$.

In the figure, the x-axis plots the initial guess angle (with a fixed initial magnitude guess of 1.0 per unit), and the y-axis is the number of iterations to converge using the NRP, with positive values indicating convergence to the OprS and negative values used to indicate AltS convergence, except -40 indicates non-convergence. Overall, 16.1% converged to the OprS, 76.1% to an AltS, and 7.8% did not converge. The ROA is centered on the OprS value of -116.6° and extends about 30° in each direction. As is the case with the 42-bus example, the flat start guess is not within the OprS's ROA.

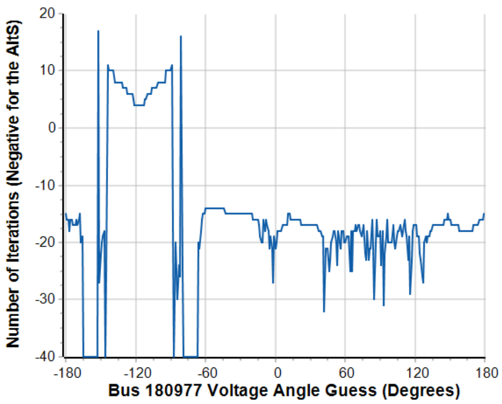

Figure 8. 23,600-Bus Electric Grid Bus 180977 Voltage Angle Convergence

The takeaway from these examples, particularly the 42-bus and 23,600-bus ones, is that with solved cases, it is relatively easy to choose an initial voltage guess with flat start values that is not in the ROA for the OprS. This could occur, for example, in the extremely common situation where a planning engineer enters new buses into a solved case and defaults to flat start voltages for the new buses. While the engineer could select other values, the goal of this paper is to present algorithm enhancements that can increase the likelihood of OpsS convergence regardless of the initial selection.

IV. Improved Solution Methods

In this section, three methods for improving the NR convergence to the OprS are considered. The first is to utilize the NRR algorithm with the optimal multiplier first proposed in [23] for the rectangular formulation, with more details given in [8] and [24]. With this approach, denoted here as the NRRO, the Newton-Raphson algorithm is modified with an optimal scalar (multiplier) that prevents divergence. The second approach is to modify the load voltage dependence so that when the load is modeled using the extremely common constant power approach, it is automatically converted into a constant current load during the solution if its voltage magnitude is below a threshold, and then into constant impedance if it is below a lower threshold. This approach, which is widely used in commercial power flow solutions, leverages the insights from [25] that noted AltSs would be unlikely when the loads' bus voltage dependence has a voltage magnitude exponent greater than unity. The third approach is the application of various heuristics that are applied before the start of the NR to modify voltage guesses that are likely to result in convergence to AltSs.

With the first approach of using the NRPO for the test cases, it does seem to minimize or perhaps eliminate the fractal aspects of the ROA. However, it doesn't seem to significantly expand the ROA. As examples, Figure 9 shows the ROA for the Figure 3 scenario, except using the NRPO, and Figure 10 shows the ROA for the Figure 5 scenario. In both cases, the ROA is not significantly expanded, now with 20.6% converging to the OprS for the two-bus scenario (compared to 20.3% previously) and 1.5% for the 42-bus case (compared to 1.0% previously). For AltS convergence with the two-bus case, essentially all of the other guesses converge to the AltS, whereas with the 42-bus case, 13.2% do, with the remainder not converging. Since the average number of iterations decreases, the NPRO is a useful approach, but is certainly not a panacea for robust OprS convergence.

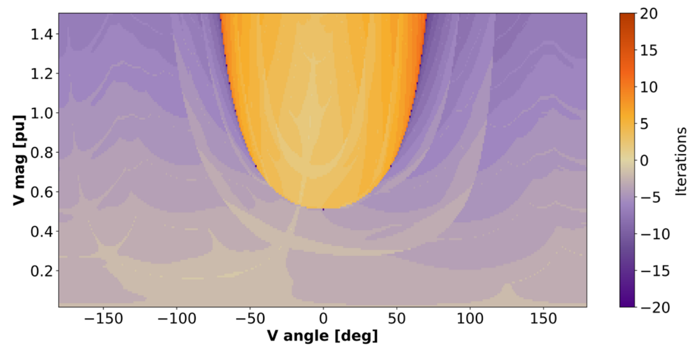

Figure 9. Two-Bus NRRO ROA Visualization

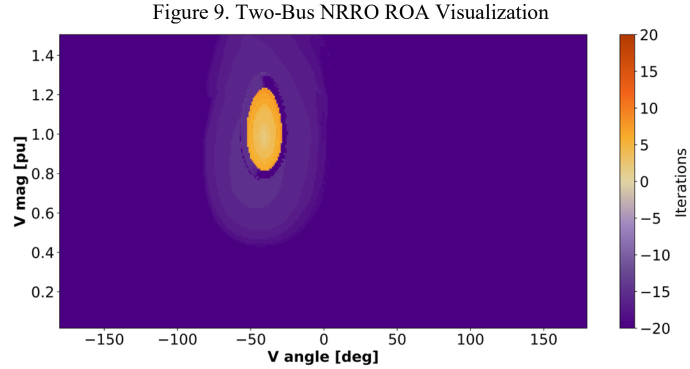

Figure 10. 42-Bus Electric Grid NRRO ROA 2D Projection for Bus 34

The second approach is to automatically convert constant power loads to constant current when their voltage magnitudes drop below a threshold (denoted as MinVP) and to constant impedance when their voltage drops below a second threshold (MinVI). Using typical values of MinVP = 0.7 pu and MinVI = 0.5 pu, for the two-bus scenario using the NRRO, the ROA, shown in Figure 11, is slightly expanded to 21.5% and has a different shape. Additionally, due to the change in the load model, the AltS now has a voltage magnitude of zero. Figure 12 uses the same parameters with the 42-bus case, with the OprS region still at 1.5%. Interestingly, and not helpful for this paper's purpose, the AltS region is now at 76.6%.

A similar result occurs in the 23,600-bus case when using MinVP = 0.7 and MinVI = 0.5, as shown in Figure 13, which displays iteration values for the 1.0 per unit voltage angle sweep. Convergence to the OptS increases from 16.1% to 20.6%, while the remainder converges to an AltS. Based on this testing, the conclusion for the second approach is that it is marginally helpful and therefore recommended, however, it still leads to many situations in which a flat start converges to an AltS.

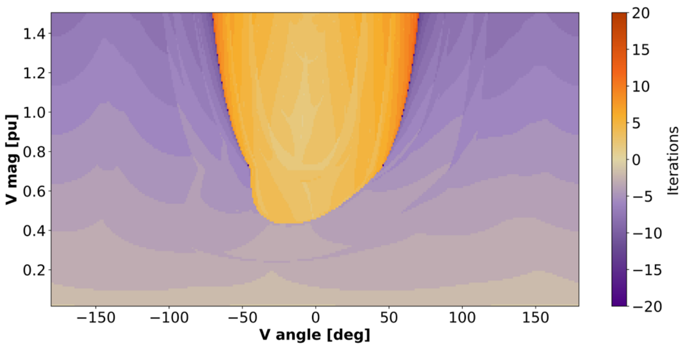

Figure 11. Two-Bus NRRO ROA with MinVP=0.7, MinVI=0.5

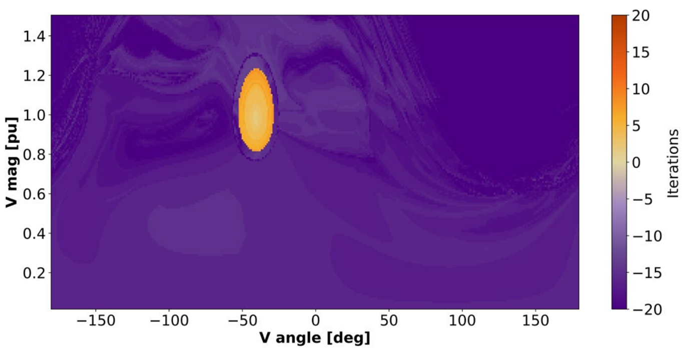

Figure 12. 42-Bus NRRO ROA with MinVP=0.7, MinVI=0.5

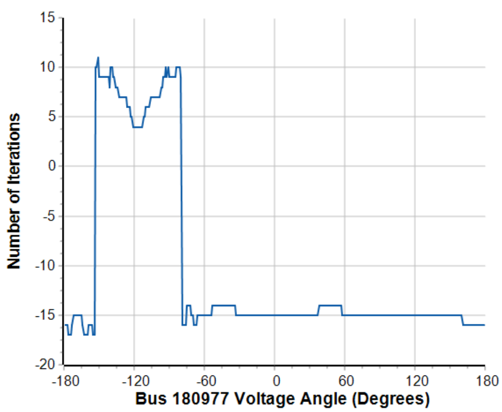

Figure 13. 23,600-Bus NRRO 1D ROA with MinVP=0.7, MinVI=0.5

The third approach is to utilize various heuristics to modify the initial guess prior to the start of the NR algorithm to avoid this situation. Again, the assumption here is the common one in practice, where an engineer takes a solved power flow and modifies it by adding new buses. While ideally the new bus voltage guesses would be close to those of their solved neighbors, there will certainly be situations in which that is not the case.

The heuristic presented here involves modifying the guesses of buses where there are large mismatches. Of course, given that the purpose of the power flow is to drive the mismatches to zero, there will be mismatches at the start of the solution. However, when reasonable voltage guesses are used, these values will usually be modest. For example, when solving a full system from a flat start at every bus, the line flows will be zero (except for differences caused by phase shifting and load-tap-changing transformers, as well as non-unity generator setpoints). The mismatches are then due to the bus power injections, which would tend to be bounded by the sum of the incident branch (i.e., the transmission lines and transformers) complex power limits. Therefore, a reasonable heuristic is to adjust the voltage guesses at buses in which the flows through the incident branches are at least somewhat above their ratings.

The balancing act in applying these heuristics is the need to catch situations that would likely result in either divergence or convergence to an AltS in a computationally efficient manner without unduly affecting the overall computational efficiency of the NR. This is achieved with the presented heuristic, as it involves calculating all the initial line flows of computational order N and then determining the mismatches and adjusting the voltages at the likely small subset of buses flagged by the heuristic. A key heuristic parameter is the amount by which the initial branch flow (calculated using the voltage guesses) exceeds its limit. If this value is too low, many branches would be flagged, whereas if it is too high, problem branches could be missed. The scalar parameter used here is 4.0, meaning that branches with initial flows above 400% of their limit are flagged for correction.

The applicability of this heuristic is considered by using it to recalculate the ROAs for the cases presented in the three previous figures. Figure 14 shows the two-bus case assuming a branch limit of 300 MVA. Now the OprS ROA is at 67.4% including all initial guesses with a 1.0 pu voltage magnitude. For the 42-bus case at Bus 34, there are two 138 kV lines, one with a rating of 250 MVA and one with a rating of 336 MVA. Figure 15 shows the 42-bus ROA for Bus 34 applying the heuristic, now with the OprS ROA at 98.6%. Last, Figure 16 shows the convergence iterations for the 23,600-bus case again at Bus 180977, now with all of them converging to the OptS.

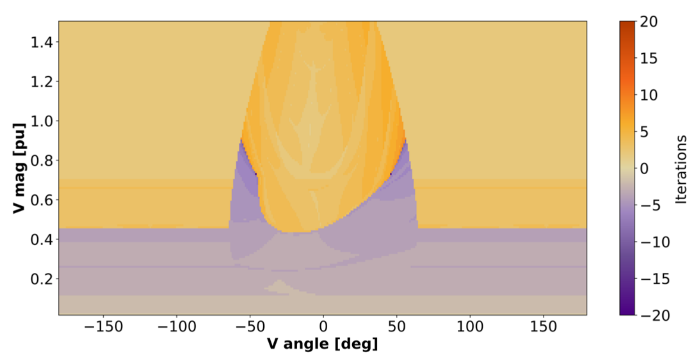

Figure 14. Two-Bus ROA Using Convergence Enhancement Heuristic

To further test this heuristic, it can be applied to flat starts at other buses. While space prevents showing more ROA visualizations, the algorithm's robustness can be assessed by just considering whether it converges to the OprS. For the 42-bus case, when testing all buses (except the slack) without the heuristic (though still with the NRPO, and with MinVP = 0.7 and MinVI = 0.5), converges to the OprS for 25 buses (61%), converges to an AltS for 32% and does not converge for the remainder. With the heuristic using a line limit scalar of 2.5, all of them converge to the OprS. For the 23,600-bus system, when testing flat start values at each bus individually without the heuristic, only 37.9% converge to the OprS. In contrast, with the presented heuristic and a line limit scalar of 2.5, this value increases to 99.0%.

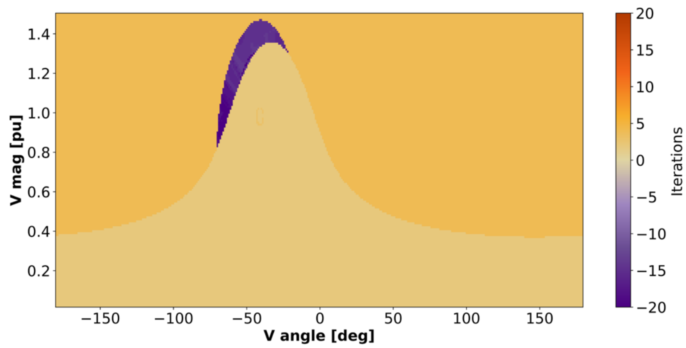

Figure 15. 42-Bus ROA Using Convergence Enhancement Heuristic

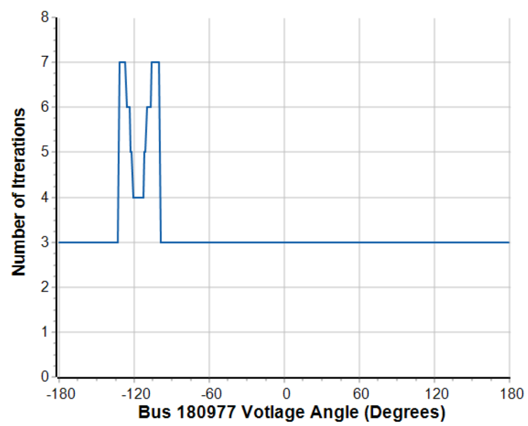

Figure 16. 23,600-Bus 1D ROA Using Convergence Enhancement Heuristic

V. Conclusion and Looking Forward

The key contributions of the paper are 1) to highlight the ease in which the NR power flow can converge to undesired AltSs, 2) to gain insight into the NR convergence by visualizing its ROAs, and 3) to test the applicability of three methods for enhancing convergence to the desired OprS. The paper demonstrates that this convergence can be significantly enhanced through the use of a simple heuristic in the common situation where flat start voltage guesses are made at a small subset of buses in an otherwise already solved power flow. Results are demonstrated on three test grids with sizes up to 23,600 buses. Future work is needed to develop additional techniques for capturing other common situations and to do further testing on realistically sized grids.

VI. Acknowledgements

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